1.2 Calculating Limits Algebraically

Standards:

MCA1

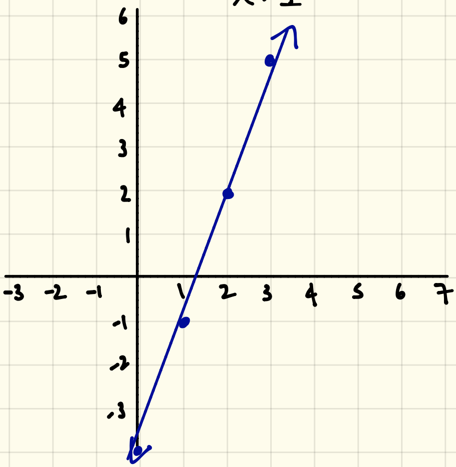
MCA1a

MCA2

MCA2a

$\boxed{\text{Old}}$ Calculating Limits Numerically & Graphically

Let's consider $\lim_{x \to 1} 3x-4$.



	x	y	
	.9	-1.3	
	.99	-1.03	
	.999	-1.003	
1	1.001	-.997	-1
	1.01	-.97	
	1.1	-.7	

So evaluating $\lim_{x \to 1} 3x-4 = -1$ because as x approaches 1 (on both sides), the y-values "tend to go" to -1.

$\boxed{\text{new}}$ Calculating Limits Algebraically

Consider $\lim_{x \to 1} 3x-4$. When we evaluate f(1), we get

$$f(1) = 3(1) - 4 = -1.$$

Conclusion $\lim_{x \to 1} 3x-4 = f(1)$.

Direct Substitution Property

Suppose $f(x)$ is a polynomial or rational function & (a) is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Direct Substitution Property:

- involves substituting (a) in for x into $f(x)$.
- is valid for all polynomials & rational function with non-zero denominators
- will work for all polynomial & rational function as long as (a) is in the domain
- Functions with this direct substitution property are called continuous at $x=a$.

[Examples]

① $\lim_{x \to 2} x-1 = (2)-1 = 1.$

② $\lim_{x \to 6} x^2-3x+7 = (6)^2-3(6)+7 = 25.$

③ $\lim_{x \to 5} \dfrac{x^2-4x+3}{x-3} = \dfrac{(5)^2-4(5)+3}{(5)-3} = \dfrac{8}{2} = 4.$

Sometimes, there are going to be times where the direct substitution property does not work initially:

(example) $\lim_{x \to 3} \frac{x^2-4x+3}{x-3} = \frac{(3)^2-4(3)+3}{3-3} = \frac{0}{0}$ ← This is not a number... what happens now?

When this form occurs $\left(\frac{0}{0}\right)$ after using the direct substitution property, we need to manipulate the limit expression to where we will be able to use the direct substitution property.

There are 4 techniques to manipulate limits:

① Factoring ② Expanding ③ Common Denominator ④ Multiply by the Conjugate

$\boxed{1}$ Factoring Method → involves factoring & eliminating.

[Example] $\lim_{x \to 3} \frac{x^2-4x+3}{x-3} = \frac{(3)^2-4(3)+3}{3-3} = \frac{0}{0}$

$$\lim_{x \to 3} \frac{x^2-4x+3}{x-3}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} x-1$$

$$=(3)-1$$

$$=2.$$

2 Expanding Method → involves expanding & simplifying.

[Example] $\lim_{h\to 0} \frac{(4+h)^2-16}{h} = \frac{(4-0)^2-16}{0} = \frac{16-16}{0} = \frac{0}{0}$

$$\lim_{h\to 0} \frac{(4+h)^2-16}{h}$$

$$= \lim_{h\to 0} \frac{16+8h+h^2-16}{h}$$

$$= \lim_{h\to 0} \frac{8h+h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(8+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 8+h$$

$$= 8+(0)$$

$$= 8.$$

$\boxed{3}$ Common Denominator → involves getting common denominator

[Example] $\lim_{h\to 0} \dfrac{\frac{1}{3+h} - \frac{1}{3}}{h} = \dfrac{\frac{1}{3+0} - \frac{1}{3}}{0} = \dfrac{0}{0}$

$$\lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{3+h}\frac{(3)}{(3)} - \frac{1}{3}\frac{(3+h)}{(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{3(3+h)} - \frac{(3+h)}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{3}-\cancel{3}-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-h}}{3(3+h)} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-1}{3(3+h)}$$

$$= \frac{-1}{3(3+0)}$$

$$= \frac{-1}{9}$$

4 Multiply by the Conjugate

[Example] $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} = \frac{\sqrt{7+2}-3}{7-7} = \frac{0}{0}$

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}$$

$$= \lim_{x\to 7} \frac{(x+2)-3\sqrt{x+2}+3\sqrt{x+2}-9}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{\cancel{x-7}}{\cancel{(x-7)}(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3}$$

$$= \frac{1}{\sqrt{7+2}+3}$$

$$= \frac{1}{3+3}$$

$$= \frac{1}{6}.$$

Absolute Value Functions

Let's consider $f(x) = |x|$.

Absolute value is a piecewise function.

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

[Example]

$$\lim_{x \to 2} \frac{|x-2|}{x-2} \rightsquigarrow |x-2| = \begin{cases} (x-2) & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2^-} -1 = -1$$

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2^+} 1 = 1.$$

So, $\lim_{x \to 2} \frac{|x-2|}{x-2} = D.N.E.$