3.2 L'Hospital's Rule

Standard:

MCA2a

$\boxed{\text{old}}$ Calculating Limits

Let's recall how to calculate limits:

- use direct substitution property
- if indetermediate form is expressed, $\frac{0}{0}$ or $\frac{\infty}{\infty}$, further algebraic manipulation is needed to be able to evaluate.

① $\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \frac{0}{0}$

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x+2 = (1)+2 = 3.$$

② $\lim_{x\to 0} \frac{(4+x)^2-16}{x} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{(4+x)^2-16}{x} = \lim_{x\to 0} \frac{(16+8x+x^2)-16}{x} = \lim_{x\to 0} \frac{8x+x^2}{x} = \lim_{x\to 0} \frac{\cancel{x}(8+x)}{\cancel{x}} = \lim_{x\to 0} 8+x$$

$= 8+(0) = 8.$

$\boxed{\text{new}}$ L'Hospital's Rule

Let's consider the rational function $\frac{f(x)}{g(x)}$.

Theorem: Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Prove that $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

proof:

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-f(a)}{\cancel{x-a}} \cdot \frac{\cancel{x-a}}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}. \blacksquare$$

[Example 1] Evaluate the limits.

① $\lim\limits_{x\to 1} \frac{x^2+x-2}{x-1} = \frac{0}{0}$

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{2x+1}{1} = \frac{2(1)+1}{1} = 3$$

② $\lim\limits_{x\to 0} \frac{(4+x)^2-16}{x} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{(4+x)^2-16}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{2(4+x)'\cdot 1 - 0}{1} = \frac{2(4+(0))'}{1} = 2(4) = 8$$

$\boxed{\text{note:}}$

Sometimes you might have to apply L'Hospital's Rule multiple times when the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ reoccurs.

③ $$\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \overset{\text{Rewrite}}{=} \lim_{x\to 0} \frac{(1+x)^{\frac{1}{2}}-1-\frac{x}{2}}{x^2}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} = \frac{-\frac{1}{4}(1+0)^{-\frac{3}{2}}}{2} = -\frac{1}{8}.$$

④ $$\lim_{x\to\infty} \frac{e^x}{x^3} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{e^x}{x^3} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \to \frac{\infty}{\infty}$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{6x} \to \frac{\infty}{\infty}$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{6} = \frac{\infty}{6} = \infty.$$

$\boxed{\text{note:}}$ If you get an indeterminate product initially, then you need to use your algebraic properties to rewrite the limit expression in order to get the indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$. Once the form, you can apply the L'H's rule.

5. $\lim\limits_{x\to\infty} x^2 e^{-x} = \infty \cdot 0$

$$\lim_{x\to\infty} x^2 e^{-x} \underline{\underline{\text{Rewrite}}} \lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$$

$$\underline{\underline{L'H}} \lim_{x\to\infty} \frac{2x}{e^x} \to \frac{\infty}{\infty}$$

$$\underline{\underline{L'H}} \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0.$$