3.5 Mean Value Theorem

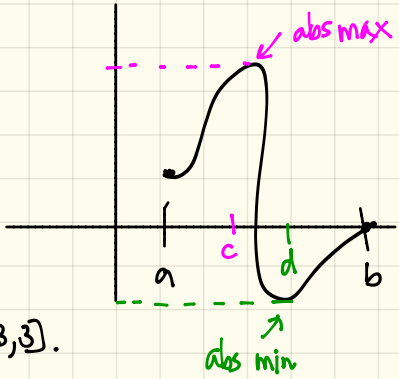
Standards:

MCD2

MCD2c

Old Extreme Value Theorem

If $f(x)$ is continuous on the closed interval $[a,b]$, then $f(x)$ attains its abs max & min somewhere on $[a,b]$.



[Example] Find abs max/min for $f(x) = x^5 - 80x$ on $[-3,3]$.

Step1: $f(x) = x^4 - 80x$

$f'(x) = 5x^4 - 80$

$5x^4 - 80 = 0$

$5(x^4 - 16) = 0$

$x^4 - 16 = 0$

$x^4 = 16$

$x = 2, -2.$

Step2/3:

Critical #'s

$f(2) = -128$

$f(-2) = 128$

Endpoints

$f(3) = 3$

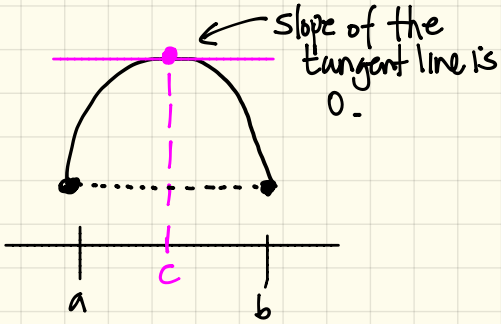
$f(-3) = -3$

Step4: The abs max is 128 occuring at $x=2$ & abs min is -128 occuring at $x=2$.

new Mean Value Theorem

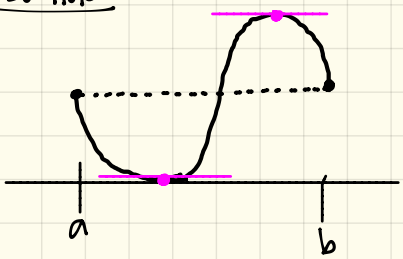
please note: The main result of this topic is the Mean Value Theorem. To prove & understand this concept, we need to know the Rolle's Theorem.

Basic Idea of Rolle's Theorem



If the endpoint values of f are equal, then f must have an extreme value at some number strictly between the endpoints, provided that f is continuous on the interval & f is differentiable, then $f'(x) = 0$.

Also note:



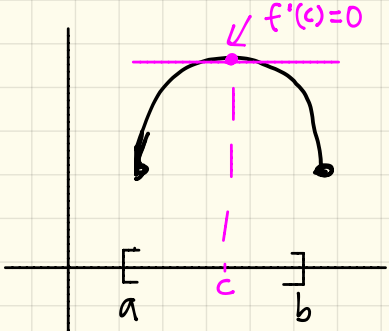
A slope might be 0 twice (or more) between numbers where the values of f are equal.

Rolle's Theorem

So suppose that

1. f(x) is continuous on [a,b]
2. f(x) is differentiable on (a,b)
3. f(a) = f(b)

Then, there must exist at least a number c in [a,b] such that $f'(c) = 0$.



Rolle's Theorem will lead us to the Mean Value Theorem.

Mean Value Theorem

So suppose that
1. $f(x)$ is continuous on $[a,b]$
2. $f(x)$ is differentiable on (a,b)

Then, there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: The Mean Value Theorem is a more general case of Rolle's Theorem which has additional hypothesis that $f(a) = f(b)$ and in conclusion states that there is a number c such that $f'(c) = 0$.

[Example 1] Find a point c satisfying the conclusion of MVT for $f(x) = \frac{1}{x}$ in $[2,8]$.

First: Is f continuous on $[2,8]$? yes

Second: Is f differentiable on $(2,8)$? yes

Now, $f'(c) = \frac{f(b) - f(a)}{b - a}$

Let $a=2$ & $b=8$. So, $\frac{f(8) - f(2)}{8 - 2} = \frac{-1}{16}$

Also, $f(c) = \frac{1}{c}$ Rewrite c^{-1}. $f'(x) = -1c^{-2} = \frac{-1}{c^2}$.

So now set $f'(c) = \frac{-1}{16}$ $\Longrightarrow$ $\frac{-1}{c^2} = \frac{-1}{16}$

$-c^2 = -16$

$c^2 = 16$

$c = \pm 4$.

But since $c = -4$ isn't in $[2,8]$, $c = -4$ is not the answer.

Therefore $c = 4$.

[Example 2] Find a point c satisfying the conclusion of MVT for $f(x)=\sqrt{x}$ on $[9,25]$.

First: Is f continuous on $[9,25]$? yes
Second: Is f is differentiable on $(9,25)$? yes

Now, $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Let $a=9, b=25$. So, $\frac{f(25)-f(9)}{25-9}=\frac{5-3}{25-9}=\frac{1}{8}$

Also $f(c)=\sqrt{c}$. Now, $f'(c)=\frac{1}{2}c^{-1/2}=\frac{1}{2\sqrt{c}}$

So, $\frac{1}{2\sqrt{c}}=\frac{1}{8}$

$8=2\sqrt{c}$
$4=\sqrt{c}$
$(4)^2=(\sqrt{c})^2$
$16=c$

Therefore, $c=16$ meeting MVT's requirements.