3.6 Chain Rule

Old Composition of Functions

Let's consider the functions: $f(x) = 2x + 1$ and $g(x) = x^3$. Perform the following tasks.

① $f \circ g(x) = f(g(x)) = 2(x^3) + 1 = 2x^3 + 1$

② $g \circ f(x) = g(f(x)) = (2x + 1)^3$

③ $f \circ f(x) = f(f(x)) = 2(2x + 1) + 1 = 4x + 2 + 1 = 4x + 3$

④ $g \circ g(x) = g(g(x)) = (x^3)^3 = x^9$

Are any of the following functions a composition of functions? If so, state the composition.

① x^3
no

② $(x^2 + 5)^3$
$f(x) = x^2 + 5$
$g(x) = x^3$

③ $\sqrt{x^3}$
$f(x) = x^3$
$g(x) = \sqrt{x}$

④ $\sin(4x)$
$f(x) = \sin(x)$
$g(x) = 4x$

⑤ $\cos^3 x$
(rewrite)
$= (\cos x)^3$
$f(x) = \cos x$
$g(x) = x^3$

⑥ $\dfrac{1}{(t^4 + 1)^3}$
$f(x) = \dfrac{1}{x^3}$
$g(x) = t^4 + 1$

⑦ $\tan(\sin x)$
$f(x) = \tan x$
$g(x) = \sin x$

⑧ $\cos(nx)$
$f(x) = nx$
$g(x) = nx$

[new] Chain Rule

We've been able to find derivatives of functions such as $f(x) = x^2$ & $g(x) = \sin(x)$. However, what if we wanted to differentiate such functions like $f(x) = \sqrt{x^2+1}$ and $g(x) = (x^3 + \tan x)^{76}$?

We need a technique to help us determine the derivative of the composition of functions.

Let's note...
Since functions $f(x) = \sqrt{x^2+1}$ & $g(x) = (x^3 + \tan x)^{76}$ are compositions, we can use a technique called the <u>chain rule</u>.

Let's consider $K(x) = f(g(x))$. Find the derivative.

$$\frac{d\,K(x)}{dx} \implies \text{need to find derivative.}$$

$$\frac{d\,K(x)}{dx} \cdot \frac{d\,g(x)}{g(x)} = \frac{d\,K(x)}{d\,g(x)} \cdot \frac{d\,g(x)}{dx} = \frac{d\,f(g(x))}{d\,g(x)} \cdot \frac{d\,g(x)}{x}$$

↑ multiply by 1.

$$= f'(g(x)) \cdot g'(x)$$

[CHAIN RULE]

$$\frac{d}{dx}\left[f(g(x))\right] = f'(\underline{g(x)}) \cdot \underline{g'(x)}$$

↑ derivative of the outside function

↑ derivative of the inside function

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

What is the outside & inside function of $f(x) = \sqrt{x^2+1}$ & $g(x) = (x^3 + \tan x)^{76}$?

[a] $f(x) = \sqrt{x^2+1}$

Outside function: $f(x) = \sqrt{x}$
inside function: $g(x) = x^2+1$

[b] $g(x) = (x^3 + \tan x)^{76}$

Outside function: $f(x) = x^{76}$
inside function: $g(x) = x^3 + \tan x$

[Examples] Find the derivatives of the functions.

① $f(x) = (x^3 + \tan x)^{76}$

$$f'(x) = 76(x^3 + \tan x)^{75} \cdot (3x^2 + \sec^2 x)$$

② $f(x) = \sqrt{x^2+1}$ Rewrite $= (x^2+1)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (2x)$$

$$= x(x^2+1)^{-\frac{1}{2}}$$
$$= \frac{x}{(x^2+1)^{\frac{1}{2}}}$$
$$= \frac{x}{\sqrt{x^2+1}}$$

③ $\dfrac{d}{dx} \sin(x^2) = \cos(x^2) \cdot (2x)$

$\qquad\qquad = 2x \cos(x^2)$

④ $f(x) = \sin^2 x$ Rewrite $(\sin x)^2$

$\qquad f'(x) = 2 \sin x \cdot \cos x$

$\qquad\qquad = 2 \sin x \cos x$

⑤ $f(x) = (x^3 - 1)^{100}$

$\qquad f'(x) = 100 (x^3 - 1)^{99} \cdot (3x^2)$

$\qquad\qquad = 300x^2 (x^3 - 1)^{99}$

⑥ $f(x) = \left(\dfrac{t-2}{2t+1}\right)^9$

$\qquad f'(x) = 9 \left(\dfrac{t-2}{2t+1}\right)^8 \cdot \left(\dfrac{(1)(2t+1) - (2)(t-2)}{(2t+1)^2}\right)$