3.8 Derivatives of Exponential Functions

Standard:

MCD1e

$\boxed{\text{Old}}$ Chain Rule

Remember to find derivatives of composition of functions, one must use the chain rule:

Chain Rule $\leadsto$ $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$.

Find the derivatives of the following:

① $f(x) = (x^2+3x)^5$

$f'(x) = 5(x^2+3x)^4 \cdot (2x)$

$= 10x(x^2+3x)^4$

② $f(x) = \sin^4 x = (\sin x)^4$

$f'(x) = 4(\sin x)^3 \cdot (\cos x)$

$= 4\sin^3 x \cos x$

③ $f(x) = \frac{1}{\sqrt{3x}} = \frac{1}{(3x)^{1/2}} = (3x)^{-1/2}$

$f'(x) = -\frac{1}{2}(3x)^{-\frac{3}{2}} \cdot (3)$

$= -\frac{3}{2}(3x)^{-\frac{3}{2}}$

$= \frac{-3}{2\sqrt{(3x)^3}}$

④ $f(x) = \sin 4x$

$f'(x) = \cos 4x \cdot (4)$

$= 4\cos 4x.$

$\boxed{\text{new}}$ Exponential Derivatives → natural e

Definition: $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Let's consider the function: $f(x) = e^x$. We are going to use the definition of the derivative to find the $\frac{d}{dx} e^x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$= \lim_{h \to 0} e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^x \cdot 1$$

$$= e^x \quad \blacksquare$$

Conclusion $\quad \dfrac{d}{dx} e^x = e^x.$

note: You will be using the chain rule a lot to find the derivatives of natural e's. DO NOT USE POWER RULE!

[Examples] Find derivatives.

① $f(x)=e^x$
$f'(x)=e^x$

② $f(x)=e^{-x}$
$f'(x)=e^{-x}\cdot(-1)$
$=-e^{-x}$

③ $f(x)=e^{2x}$
$f'(x)=e^{2x}\cdot(2)$
$=2e^{2x}$

④ $f(x)=e^{x+x^2}$
$f'(x)=e^{x+x^2}\cdot(1+2x)$
$=(1+2x)e^{x+x^2}$

⑤ $f(x)=2e^x$
$f'(x)=2e^x.$

more new Exponential Derivatives → exponential form

Recall an exponential function is where the base number is fixed and the exponent is the variable

$f(x)=a^x$, where $a\neq 0$.

Examples) $5^x, 2^x, \left(\frac{1}{2}\right)^x$

Not → $x^2, x^3, x^{1/2}$
(power functions or polynomials)

Let's consider the function: $f(x) = a^x$, where $a \neq 0$.
We want the derivative.

1st – Let's write a^x in terms of $e^x \longrightarrow a^x = e^{\ln a^x}$

2nd – Find the derivative using the rewrite:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln a^x}\right) = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a} \cdot (\ln a) = a^x (\ln a).$$

Exponential Rules for Derivatives

$\frac{d}{dx}(e^x) = e^x$ $\qquad$ $\frac{d}{dx}(a^x) = a^x \cdot (\ln a)$

note: ALWAYS use chain rule on exponent!

[Examples]

① $\frac{d}{dx}(8^x) = 8^x \cdot \ln 8$

② $f(x) = 7^{-x}$

$f'(x) = 7^{-x} \cdot \ln(7) \cdot (-1)$

$= -7^{-x} \ln 7.$

③ $f(x) = 10^{\sin x}$

$f'(x) = 10^{\sin x} \cdot \ln(10) \cdot (\cos x)$

$= \cos x \, 10^{\sin x} \ln 10.$