

5.2 Area Between Curves

Standards:

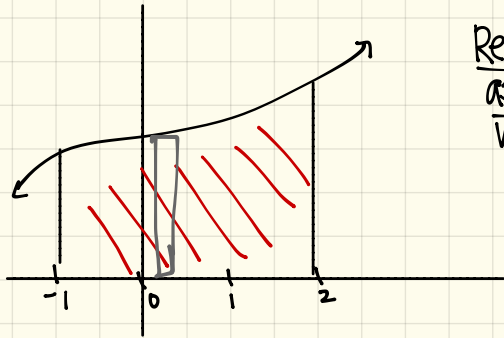
MC11

MC11c



Old] Finding Area using Integration

Let's consider the function $f(x) = x^3 + 2$. Find the area between $x = -1$ & $x = 2$.

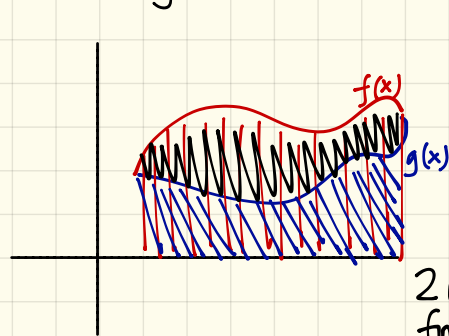


Recall: we can find the area either by approximating with rectangles or computing with Integrals.

$$\int_{-1}^2 x^3 + 2 \, dx = \left[\frac{x^4}{4} + 2x \right]_{-1}^2 = \left[\frac{(2)^4}{4} + 2(2) \right] - \left[\frac{(-1)^4}{4} + 2(-1) \right] = [8] - [-1.75]$$
$$= 9.75.$$

New] Area between Curves

Let's consider 2 functions: $f(x)$ & $g(x)$. We want to find the area between 2 arbitrary curves.



$$\int_a^b f(x) \, dx - \int_a^b g(x) \, dx =$$

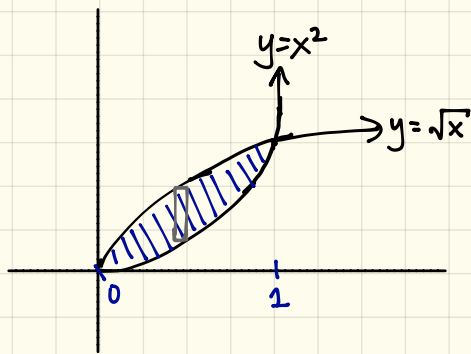
Area under
2 curves
formula
with respect
to x .

$$\int_a^b f(x) - g(x) \, dx$$

the most
top curve

the most
bottom curve

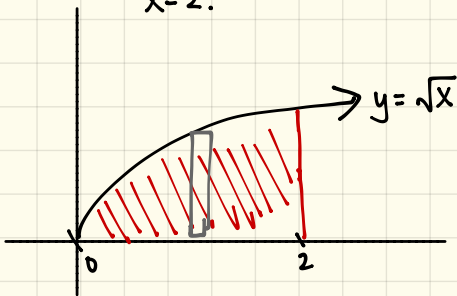
[Example 1] Find the area under the curves.



$$\begin{aligned} \int_0^1 [\sqrt{x}] - [x^2] dx &= \int_0^1 x^{1/2} - x^2 dx \\ &= \left[\frac{2}{3} x^{3/2} - \frac{x^3}{3} \right]_0^1 = \left[\frac{2}{3} (1)^{3/2} - \frac{(1)^3}{3} \right] - [0] \\ &= \frac{2}{3} - \frac{1}{3} = \left(\frac{1}{3} \right) \end{aligned}$$

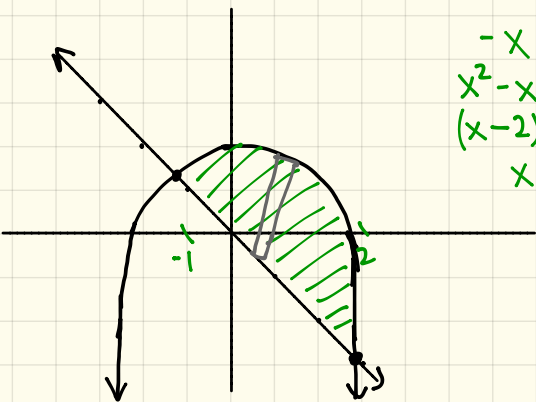
$$\begin{aligned} (\sqrt{x})^2 &= (x^2)^2 \\ x &= x^4 \\ 0 &= x^4 - x \\ 0 &= x(x^3 - 1) \\ x &= 0, 1 \end{aligned}$$

[Example 2] Find the area between the curves $y=0$ & $y=\sqrt{x}$ between $x=0$ and $x=2$.



$$\begin{aligned} \int_0^2 \sqrt{x} - 0 dx &= \int_0^2 x^{1/2} dx = \frac{2}{3} x^{3/2} \\ &= \left[\frac{2}{3} \sqrt{x^3} \right]_0^2 = \left[\frac{2}{3} \sqrt{(2)^3} \right] - \left[\frac{2}{3} \sqrt{0^3} \right] \\ &= \frac{2\sqrt{8}}{3} \end{aligned}$$

[Example 3] Find the area between $y = -x$ & $y = 2 - x^2$.



$$\begin{aligned} -x &= 2 - x^2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2, -1 \end{aligned}$$

$$\int_{-1}^2 [2 - x^2] - [-x] dx = \int_{-1}^2 2 - x^2 + x dx = 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^2$$

$$= \left[2(2) - \frac{(2)^3}{3} + \frac{(2)^2}{2} \right] - \left[2(-1) - \frac{(-1)^3}{3} + \frac{(-1)^2}{2} \right]$$

$$= \left[4 - \frac{8}{3} + \frac{4}{2} \right] - \left[-2 + \frac{1}{3} - \frac{1}{2} \right]$$

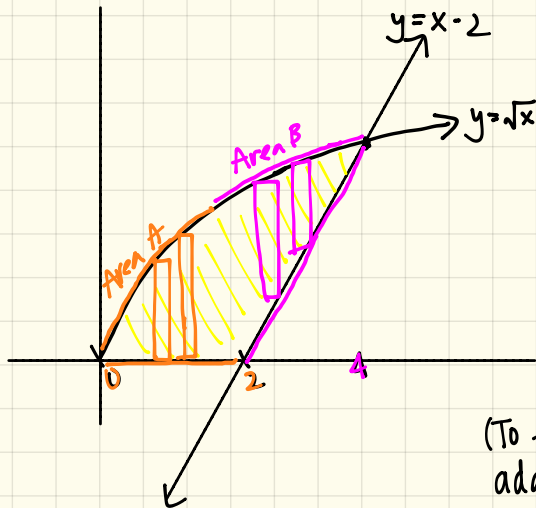
$$= 4 + 2 - \frac{8}{3} - \frac{1}{3} + \frac{4}{2} - \frac{1}{2}$$

$$= 6 - \frac{9}{3} + 2 - \frac{1}{2}$$

$$= 6 - 3 + 2 - \frac{1}{2}$$

$$= 5 - \frac{1}{2} = \textcircled{4.5}$$

[Example 4] Find the area of curves bounded by $y=\sqrt{x}$, $y=0$ and $y=x-2$.



Area A + Area B

$$\text{Area A} = \int_0^2 [\sqrt{x}] - [0] dx$$

$$\text{Area B} = \int_2^4 [\sqrt{x}] - [x-2] dx$$

o
o
o

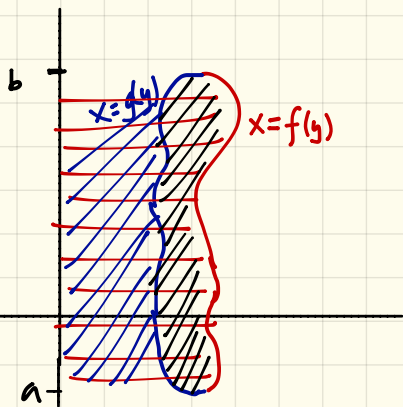
Do on your own.

(To finish, evaluate both integrals & add the results.)

$$\begin{aligned}x-2 &= 0 \\ x &= 2\end{aligned}$$

$$\begin{aligned}(\sqrt{x})^2 &= (x-2)^2 \\ x &= (x-2)^2 \\ x &= x^2 - 4x + 4 \\ 0 &= x^2 - 5x + 4 \\ 0 &= (x-4)(x-1) \\ 0 &= x-4, \cancel{x}\end{aligned}$$

Now let's consider the 2 equations: $x = f(y)$ and $x = g(y)$.



$$\int_a^b f(y) dy - \int_a^b g(y) dy =$$

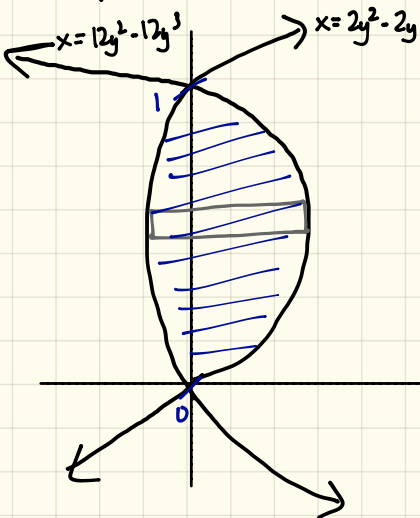
$$= \int_a^b f(y) - g(y) dy$$

Area between 2
the 2 curves
formula in
respect to y

the most
right curve

the most
left curve

[Example 1] Find the area between curves.



$$\begin{aligned} & \int_0^1 [(12y^2 - 12y^3) - (2y^2 - 2y)] dy \\ &= \int_0^1 [12y^2 - 12y^3 - 2y^2 + 2y] dy \\ &= \left[\frac{12y^3}{3} - \frac{12y^4}{4} - \frac{2y^3}{3} + \frac{2y^2}{2} \right]_0^1 \\ &= \left[4y^3 - 3y^4 - \frac{2}{3}y^3 + y^2 \right]_0^1 \\ &= [4(1)^3 - 3(1)^4 - \frac{2}{3}(1)^3 + (1)^2] - [0] \\ &= 4 - 3 - \frac{2}{3} + 1 \\ &= 2 - \frac{2}{3} = \frac{6}{3} - \frac{2}{3} = \left(\frac{4}{3} \right) \end{aligned}$$